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NON-CONJUGACY OF A MINIMAL DISTAL DIFFEOMORPHISM OF THE TORUS TO A C^1 SKEW-PRODUCT

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ABSTRACT

An example is given of a positively oriented minimal distal C^∞ diffeomorphism of the torus which is not topologically conjugate to a C^1 skew-product.

Introduction

Let $\mathrm{Hom}^+(K^n)$ denote the set of orientation-preserving homeomorphisms of the n-dimensional torus $K^n = \mathbf{R}^n/\mathbf{Z}^n$. If T is a minimal element of $\mathrm{Hom}^+(K)$, then it is known that T is topologically conjugate to an irrational rotation of K, which is, of course, C^∞. Correspondingly, if T is a minimal *distal* element of $\mathrm{Hom}^+(K^2)$, it is known (see, for instance, [4]) that T is topologically conjugate to a homeomorphism of K^2 of the form:

$$T_{\alpha,g}: (x, y) \mapsto (x + \alpha, y + g(x)) \quad \text{where } g \in C(K, K) \text{ and } \alpha \text{ is irrational.}$$

In this paper, it is shown that, contrary to what happens for the circle, or for almost periodic homeomorphisms in general, there is a minimal distal C^∞ element of $\mathrm{Hom}^+(K^2)$ which is not topologically conjugate to any C^1 homeomorphism of the form $T_{\alpha,g}$.

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§1. Preliminaries

1.1. If $f \in C(K^n, K^n)$, then there exists a unique element of $C(\mathbf{R}^n, \mathbf{R}^n)$, again denoted by f, such that $f(\mathbf{0}) \in [0, 1)^n$, and the following diagram commutes:

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$$\begin{array}{ccccc} x & \mathbf{R}^n & \xrightarrow{\quad f \quad} & \mathbf{R}^n & x \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ \mathbf{Z}^n + x & K^n & \xrightarrow{\quad f \quad} & K^n & \mathbf{Z}^n + x \end{array}$$

For $\mathrm{Hom}^+(K)$, this correspondence reduces to a correspondence between $\mathrm{Hom}^+(K)$ and $\{f \in C(\mathbf{R}, \mathbf{R}) : f$ is a homeomorphism, $f(x+1) = f(x) + 1$ for all $x \in \mathbf{R}$, and $f(0) \in [0, 1)\}$.

Note that in what follows, for all equations (inequalities) involving elements of $C(\mathbf{R}^n, \mathbf{R}^n)$ corresponding to elements of $C(K^n, K^n)$, the equality (inequality) sign denotes *real* equality (inequality) and *not* equality (inequality) mod $\mathbf{Z}^n$.

1.2. Let $\mathrm{Hom}^+(K)$ be given the topology of uniform convergence. The *rotation number function* $\rho : \mathrm{Hom}^+(K) \to K$ is continuous.

If $q \in \mathbf{Z}$ and $f \in \mathrm{Hom}^+(K)$, then $\rho(f) = \mathbf{Z} + (p/q)$ for some $p \in \mathbf{Z}$ if and only if there exists $x \in K$ with $f^q(x) = x$. (See, for example, [3], [1] for definition and basic properties of ρ.)

1.3. Definition. Let $f \in \mathrm{Hom}^+(K)$ with $\rho(f) = \mathbf{Z} + (p/q)$ with p, q coprime and positive, $0 \leqq p < q$. We follow [1] in defining f to be *semistable forward* if:

$$f^q(x) \geqq x + p \qquad \text{for all } x \in \mathbf{R}.$$

1.4. Denjoy's Theorem. (See, for example, [3].) *Let $f \in \mathrm{Hom}^+(K)$ be C^2 and $\rho(f) = \mathbf{Z} + \alpha$, $\alpha \in [0, 1)$ and irrational. Then there exists a unique $\varphi \in \mathrm{Hom}^+(K)$ such that*:

$$\varphi(f(x)) = \varphi(x) + \alpha \qquad \textit{for all } x \in \mathbf{R}, \quad \varphi(0) = 0.$$

φ is called the *eigenfunction of f corresponding to α*. Note that, in particular, f is minimal almost periodic.

§2. Reduction of the problem

Throughout this section, let $f \in \mathrm{Hom}^+(K)$ be C^∞ with $\rho(f) = \mathbf{Z} + \alpha$, α irrational, $\alpha \in [0, 1)$.

Let $T \in \mathrm{Hom}^+(K^2)$ be given by:

$$T(x, y) = (f(x), x + y).$$

Then (K^2, T) is distal, and the maximal almost periodic factor is (K, f). Since (K, f) is minimal by 1.4, (K^2, T) is minimal by [2] §2.

Consider the following four statements. It will be shown that 2.4 $\Rightarrow$ 2.3 $\Rightarrow$ 2.2 $\Rightarrow$ 2.1.

2.1. If $T(x,y) = (f(x), x+y)$, then T is not conjugate to any C^1 homeomorphism of the form:

$$T_{\beta,g} : (x,y) \mapsto (x+\beta, y+g(x)), \qquad \text{where } \beta \in \mathbf{R} \text{ and } g \in C^1(K,K).$$

2.2. The equation:

$$x - \varphi(x) = \psi(\varphi(x)) + \chi(f(x)) - \chi(x) + \mu$$

does not hold for any $\psi \in C^1(\mathbf{R},\mathbf{R})$, $\chi \in C(\mathbf{R},\mathbf{R})$, $\mu \in \mathbf{R}$, where ψ and χ have period 1, $\int_0^1 \psi = 0$, and φ is the eigenfunction of f corresponding to α (see 1.4).

2.3. For each $\psi \in C^1(\mathbf{R},\mathbf{R})$ with period 1 and $\int_0^1 \psi = 0$, there exists a strictly increasing sequence $\{m_n\}$ of positive integers with:

(i) $$\sup_n \sup_{x\in\mathbf{R}} \left| \sum_{i=0}^{m_n-1} \psi(x+i\alpha) \right| < \infty.$$

(ii) The sequence

$$\left\{ \sup_{x\in\mathbf{R}} \left| \sum_{i=0}^{m_n-1} (f^i(x) - i\alpha) - (m_n/m_{n+1}) \sum_{i=0}^{m_{n+1}-1} (f^i(x) - i\alpha) \right| \right\}$$

is unbounded.

2.4. There exists a constant $B > 0$, a sequence $\{q_n\}$ of postiive integers with $q_{n+1} > q_n^6$ and a sequence $\{x_n\}$ of elements of $\mathbf{R}$ such that, if for each n, m_n is any multiple of q_n with $q_n \leqq m_n \leqq q_n^2$, then:

(i) $$\left| \frac{1 - e^{2\pi i r m_n \alpha}}{1 - e^{2\pi i r \alpha}} \right| \leqq 1 \text{ for } r \leqq q_n^6,\ r \text{ not a multiple of } q_n.$$

(ii) $$\left\{ (1/m_{n+1}) \sum_{i=0}^{m_{n+1}-1} (f^i(x_n) - i\alpha) \right\} - \left\{ (1/m_n) \sum_{i=0}^{m_n-1} (f^i(x_n) - i\alpha) \right\} \geqq B/q_n.$$

2.2 $\Rightarrow$ 2.1. If T is conjugate to a C^1 homeomorphism of the form $T_{\beta,g}$, we can assume $\beta = \alpha$, and that the conjugacy is given by:

> $(x,y) \mapsto (\varphi(x), h(x) + y)$ where $h \in C(K,K)$ and φ is the eigenfunction of f corresponding to α.

This is essentially because the group of eigenvalues is preserved under conjugacy, and a conjugacy must give 1–1 correspondences between the groups of eigenfunctions, and between the groups of generalized eigenfunctions of order 2. The result follows.

2.3 ⇒ 2.2. Suppose 2.2 does not hold, i.e. the equation of 2.2 is satisfied by some ψ, χ, μ. Replacing x by $f^i(x)$ in the equation, we obtain:

$$f^i(x)-i\alpha-\varphi(x)-\mu=\psi(\varphi(x)+i\alpha)+\chi(f^{i+1}(x))-\chi(f^i(x)).$$

Summing over i from 0 to m_n-1, we obtain:

$$\sum_{i=0}^{m_n-1}(f^i(x)-i\alpha)-m_n\varphi(x)-m_n\mu=\sum_{i=0}^{m_n-1}\psi(\varphi(x)+i\alpha)+\chi(f^{m_n}(x))-\chi(x).$$

Then (i) and (ii) of 2.3 cannot hold simultaneously for any sequence $\{m_n\}$.

2.4 ⇒ 2.3. Suppose 2.4 holds.

Let $\psi\in C^1(\mathbf{R},\mathbf{R})$ have period 1, and $\int_0^1\psi=0$. It sufficces to find a sequence $\{m_n\}$ with $q_n\leqq m_n\leqq q_n^2$, m_n a multiple of q_n, such that $\{m_n/q_n\}$ is unbounded, and:

$$\sup_n\sup_{x\in\mathbf{R}}\left|\sum_{i=0}^{m_n-1}\psi(x+i\alpha)\right|<\infty.$$

Suppose $\psi(x)=\sum_{r=-\infty}^{\infty}a_re^{2\pi irx}$. Then $\Sigma|a_r|^2r^2<\infty$ and $a_0=0$. For all $x\in\mathbf{R}$, $\sum_{i=0}^{m_n-1}\psi(x+i\alpha)=\sum_{r=-\infty}^{\infty}a_r(\sum_{s=0}^{m_n-1}e^{2\pi irs\alpha})e^{2\pi irx}$. For each r, $|\sum_{s=0}^{m_n-1}e^{2\pi irs\alpha}|\leqq m_n$. So

$$\sum_{r=-\infty}^{\infty}|a_r|\left|\sum_{s=0}^{m_n-1}e^{2\pi irs\alpha}\right|\leqq\sum_{|r|\leqq m_n^3}{}'|a_r|\left|\frac{e^{2\pi irm_n\alpha}-1}{e^{2\pi ir\alpha}-1}\right|+\sum_{|r|>m_n^3}r^{1/3}|a_r|$$
$$+m_n\sum_{t=-q_n^5}^{q_n^5}|a_{tq_n}|,$$

where Σ' denotes that the r th term is omitted if r is a multiple of q_n. Then, by 2.4(i):

$$\left|\sum_{i=0}^{m_n-1}\psi(x+i\alpha)\right|\leqq\sum_{-\infty}^{\infty}r^{1/3}|a_r|+m_n\sum_{|t|\geqq1}|a_{tq_n}|;$$

$$\sum_{r=-\infty}^{\infty}r^{1/3}|a_r|\leqq\left\{\sum_{r=-\infty}^{\infty}r^{-4/3}\right\}^{1/2}\times\left\{\sum_{r=-\infty}^{\infty}|a_r|^2r^2\right\}^{1/2}<\infty.$$

Thus it suffices to find a sequence $\{m_n\}$ such that:

(2.5) $\{m_n/q_n\}$ is unbounded, $q_n\leqq m_n\leqq q_n^2$, m_n is a multiple of q_n and:

$$\sup_n m_n\sum_{|t|\geqq1}|a_{tq_n}|<\infty.$$

Now

$$\sum_{|t|\geqq 1} |a_{tq_n}| \leqq \left\{ \sum_{|r|\geqq q_n} r^2 |a_r|^2 \right\}^{1/2} \times \left\{ \sum_{|t|\geqq 1} (1/(t^2 q_n^2)) \right\}^{1/2}.$$

Write $C = \{\Sigma_{|t|\geqq 1}(1/t^2)\}^{1/2}$ and $\gamma(q_n) = \{\Sigma_{|r|\geqq q_n} r^2 |a_r|^2\}^{1/2}$. Then $\gamma(q_n) \to 0$ as $n \to \infty$ and $\Sigma_{|t|\geqq 1} |a_{tq_n}| \leqq C\gamma(q_n)/q_n$.

Now take m_n to be the greatest multiple of q_n which is not greater than $\mathrm{Min}(q_n/\gamma(q_n), q_n^2)$, or take $m_n = q_n$ if q_n is too small for such a multiple to exist. Then the sequence $\{m_n\}$ satisfies (2.5), as required.

§3. Solution of the reduced problem

We are now reduced to constructing a C^∞ $f \in \mathrm{Hom}^+(K)$ with $\rho(f) = \alpha$, α irrational, such that f, α satisfy the conditions of 2.4. The construction is similar to Arnold's construction [1] of a C^∞ $f \in \mathrm{Hom}^+(K)$ with irrational rotation number and eigenfunction which is not absolutely continuous.

The construction of f. Sequences $\{f_n\}$, $\{p_n\}$, $\{q_n\}$, $\{x_n\}$ $(n \geqq 1)$ will be constructed such that:

3.1. Each f_n is defined and analytic in $\{z : |\mathrm{im}\, z| < 1\}$, $f_n(\mathbf{R}) \subseteq \mathbf{R}$, $f_n(z+1) = f_n(z) + 1$ for all z, $f_n(0) \in [0, 1)$, $f_n'(x) > 1/2$ for all $x \in \mathbf{R}$, (so that $f_n \mid \mathbf{R} \in \mathrm{Hom}^+(K)$), $f_{n+1} \mid \mathbf{R} \geqq f_n \mid \mathbf{R}$ and:

$$\sup_{|\mathrm{Im}\, z| < 1} |f_n(z) - f_{n+1}(z)| < 1/2^n.$$

3.2. p_n and q_n are coprime, $0 < p_n < q_n$, $\rho(f_n) = \mathbf{Z} + (p_n/q_n)$, $q_{n+1} > q_n^6$ and $p_{n+1}/q_{n+1} - p_n/q_n = 1/q_n q_{n+1}$.

3.3. f_n is semistable forward and has exactly one *cycle*, i.e. exactly one finite minimal f_n-invariant set (see 1.2).

3.4–3.6 hold for any sequence $\{m_n\}$ of positive integers such that m_n is a multiple of q_n with $q_n \leqq m_n \leqq q_n^2$:

3.4. $$\left| \left| \frac{1 - e^{2\pi i r m_s p_n/q_n}}{1 - e^{2\pi i r p_n/q_n}} \right| - \left| \frac{1 - e^{2\pi i r m_s p_{n+1}/q_{n+1}}}{1 - e^{2\pi i r p_{n+1}/q_{n+1}}} \right| \right| < 1/2^n,$$

for $r \leqq q_s^6$, r not a multiple of q_s, $s \leqq n$.

3.5. $$\sup_{x \in \mathbf{R}} \left| (1/m_r) \sum_{i=0}^{m_r - 1} (f_n^i(x) - ip_n/q_n) - (1/m_r) \sum_{i=0}^{m_r-1} (f_{n+1}^i(x) - ip_{n+1}/q_{n+1}) \right| < (1/2^{n+4})q_r \qquad \text{for } r \leqq n.$$

3.6. $\{x_n\}$ is a sequence in $\mathbf{R}$ and:

$$(1/m_{n+1})\sum_{i=0}^{m_{n+1}-1}(f_{n+1}^i(x_n)-ip_{n+1}/q_{n+1})>(1/m_n)\sum_{i=0}^{m_n-1}(f_n^i(x_n)-ip_n/q_n)+(1/4q_n).$$

Then let $f=\lim_n f_n$, $\alpha=\lim_n p_n/q_n$.

3.2 implies $|p_n/q_n-p_{n+1}/q_{n+1}|<1/n^2q_n^2$ for sufficiently large n, hence α is irrational ([1] §1).

Taking limits in 3.4–3.6 implies f, α satisfy 2.4, with $B=1/8$ in 2.4(ii). For 3.5 implies that:

$$\left|(1/m_r)\sum_{i=0}^{m_r-1}(f_r^i(x)-ip_r/q_r)-(1/m_r)\sum_{i=0}^{m_r-1}(f^i(x)-i\alpha)\right|<1/16q_r.$$

Now use this in 3.6 with $r=n$ and $r=n+1$, to get 2.4(ii) with $B=1/8$.

Let p_1, q_1 be arbitrary coprime integers, $0<p_1<q_1$, and take any f_1 satisfying 3.1 and 3.3 with $\rho(f_1)=p_1/q_1+\mathbf{Z}$. (Use [1] §1 lemma α to get a unique cycle for f_1.)

Suppose f_n, p_n, q_n have been chosen and define x_n, f_{n+1}, p_{n+1}, q_{n+1} as follows:

Choice of x_n. There are precisely q_n points in any half-open interval of $\mathbf{R}$ of length one, which correspond to the points of the unique cycle of $f_n\,|\,\mathbf{R}\in\mathrm{Hom}^+(K)$. Let $y,z\in\mathbf{R}$ correspond to points in the cycle with $y<z$, and such that if $y<w<z$, then w does not correspond to a point in the cycle. Then for each i, $f_n^i(y)$ and $f_n^i(z)$ have the same property.

Choose x_n with $y<x_n<z$ and such that:

$$0<f_n^i(x_n)-f_n^i(y)<(1/8)(f_n^i(z)-f_n^i(y)),\qquad 0\leqq i\leqq q_n^2-1.$$

Then if m_n is any multiple of q_n with $q_n\leqq m_n\leqq q_n^2$:

$$(3.7)\quad (1/m_n)\sum_{i=0}^{m_n-1}(f_n^i(z)-f_n^i(x_n))>(7/8m_n)\sum_{i=0}^{m_n-1}(f_n^i(z)-f_n^i(y))=7/(8q_n).$$

LEMMA. $(1/s)\sum_{i=0}^{s-1}(f_n^i(x_n)-ip_n/q_n)\to(1/q_n)\sum_{i=0}^{q_n-1}(f_n^i(z)-ip_n/q_n)$ *as* $s\to\infty$.

PROOF. Clearly, it suffices to show:

$$(1/rq_n)\sum_{i=0}^{rq_n-1}(f_n^i(x_n)-ip_n/q_n)\to(1/q_n)\sum_{i=0}^{q_n-1}(f_n^i(z)-ip_n/q_n)\qquad\text{as } r\to\infty.$$

But

$$(1/rq_n)\sum_{i=0}^{rq_n-1}(f_n^i(x_n)-ip_n/q_n)=(1/q_n)\sum_{i=0}^{q_n-1}\left\{(1/r)\sum_{s=0}^{r-1}(f_n^{i+sq_n}(x_n)-sp_n-ip_n/q_n)\right\}.$$

So it suffices to show that for each i, $0 \leqq i \leqq q_n - 1$:

$$(1/r) \sum_{s=0}^{r-1} (f_n^{i+sq_n}(x_n) - sp_n - i(p_n/q_n)) \to f_n^i(z) - ip_n/q_n \qquad \text{as } r \to \infty.$$

For this it suffices to show:

$$f_n^{i+sq_n}(x_n) - sp_n - ip_n/q_n \to f_n^i(z) - ip_n/q_n \qquad \text{as } s \to \infty.$$

But this follows from there being no elements of the cycle of f_n between x_n and z ([1] §1). Q.E.D.

Now choose $t_n > q_n^6$ such that:

$$\left| (1/t) \sum_{i=0}^{t-1} (f_n^i(x_n) - ip_n/q_n) - (1/q_n) \sum_{i=0}^{q_n-1} (f_n^i(z) - ip_n/q_n) \right| < 1/(8q_n)$$

for all $t \geqq t_n$. Then if $t \geqq t_n$:

$$\begin{aligned}(1/t) \sum_{i=0}^{t-1} (f_n^i(x_n) - ip_n/q_n) &> (1/q_n) \sum_{i=0}^{q_n-1} (f_n^i(z) - ip_n/q_n) - 1/(8q_n) \\ &= (1/m_n) \sum_{i=0}^{m_n-1} (f_n^i(z) - ip_n/q_n) - 1/(8q_n) \\ &> (1/m_n) \sum_{i=0}^{m_n-1} (f_n^i(x_n) - ip_n/q_n) + 3/(4q_n)\end{aligned} \tag{3.8}$$

by (3.7), where m_n is any multiple of q_n with $q_n \leqq m_n \leqq q_n^2$.

Choice of p_{n+1}, q_{n+1}. Choose $1/2^n > \delta_n > 0$ such that if $0 < \lambda < \delta_n$, $f_{n+1}(z) = f_n(z) + \lambda$ ($|\operatorname{im} z| < 1$), and f_{n+1} is semistable forward with rotation number p_{n+1}/q_{n+1}, then f_{n+1}, p_{n+1}, q_{n+1} satisfy conditions 3.4, 3.5. Choose $a, b \in \mathbf{Z}$ such that $aq_n - bp_n = 1$.

Take $q_{n+1} = b + uq_n$, $p_{n+1} = a + up_n$, for u large enough to ensure $q_{n+1} \geqq t_n$, and such that $\rho(f_n + \delta_n) > p_{n+1}/q_{n+1}$. Then p_{n+1}, q_{n+1} satisfy 3.2 and 3.4.

Choice of f_{n+1}. Suppose $\rho(f_n + \lambda_n) = p_{n+1}/q_{n+1}$, where $f_n + \lambda_n$ is semistable forward. Such a λ_n exists and is unique ([1] §1).

Choose $f_{n+1}(z) = f_n(z) + \lambda_n + \varepsilon_n(z)$ such that $\rho(f_{n+1}) = p_{n+1}/q_{n+1}$, $\varepsilon_n(x) \geqq 0$ for all $x \in \mathbf{R}$, f_{n+1} has a unique cycle, and ε_n is small enough to ensure 3.1–3.5 are satisfied ([1] §1).

Verification that 3.6 *is satisfied.*

$$(1/m_{n+1}) \sum_{i=0}^{m_{n+1}-1} (f_{n+1}^i(x_n) - ip_{n+1}/q_{n+1}) \geqq (1/q_{n+1}) \sum_{i=0}^{q_{n+1}-1} (f_{n+1}^i(x_n) - ip_{n+1}/q_{n+1})$$

(since f_{n+1} is semistable forward)

$$\geqq (1/q_{n+1}) \sum_{i=0}^{q_{n+1}-1} (f_n^i(x_n) - ip_n/q_n) - (1/q_{n+1}) \sum_{i=0}^{q_{n+1}-1} (ip_{n+1}/q_{n+1} - ip_n/q_n)$$

$$> (1/m_n) \sum_{i=0}^{m_n-1} (f_n^i(x_n) - ip_n/q_n) + 3/(4q_n) - 1/(2q_n),$$

by 3.8, 3.2 and because $q_{n+1} \geqq t_n$, where m_n and m_{n+1} are multiples of q_n, q_{n+1} respectively.

The construction is completed.

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